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PROPERTIES OF POLYGONS.

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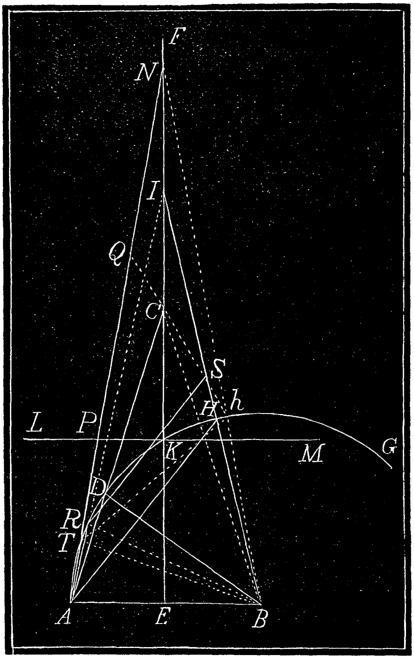
In the following article I propose to point out some interesting relations that exist between certain lines and areas of polygons, that, so far as I know, have never heretofore been announced; and to present a method for the construction of certain polygons that have never been constructed geometrically, which, though not strictly geometrical, yet from the analogy of the construction to strictly geometrical constructions I am induced to believe that geometrical constructions for these polygons may yet be found.



1. Draw the line $A\ B =$ the unit of our scale, and bisect it in E, and at E erect the indefinite perpendicular $E\ F$. With $A\ B$ as radius describe the arc $A\ G$, and draw the straight line $A\ C$ cutting the arc $A\ G$ in D, so that $D\ C$ shall equal $A\ B$. Then is $A\ B$ one side of a *pentagon* inscribed in a circle which passes through the three points A, B and C.

2. Draw the straight line $B\ I$ cutting the arc $A\ G$ in H so that $H\ I = A\ H$. Then is $A\ B$ one side of a *heptagon* inscribed in a circle which passes through the three points A, B and I.

3. Through K, the point where the arc $A\ G$ cuts the perpendicular $E\ F$, draw the line $L\ M$ parallel to $A\ B$, and draw the straight line $A\ N$ cutting $L\ M$ in P so that $P\ N$ shall equal twice $A\ B$. Then is $A\ B$ one

side of a *nonagon* inscribed in a circle which passes through the three points A, B and N.

The demonstration of these constructions is easy and is therefore omitted for the sake of brevity.

If in the heptagon whose sides each equals *one*, we put the chord of the arc which contains three of the equal sides $= 2 + x$, then will $2 + x$ be the radius of a circle in which if a polygon of double the number of sides (14) be described each of these fourteen equal sides will $=$ *one*; and the length of one side of a heptagon described in a circle whose radius is one will be $\sqrt{1 - x}$.

Also, if in the nonagon each of whose sides equals *one* we put the chord of the arc which contains three of the equal sides $= 2 + x$, then will one side of the nonagon described in a circle whose radius is one be $\sqrt{1 - x}$. And in general, if in a polygon of n equal sides (n being any number greater than six) each of which equals *one*, we put $2 + x =$ the chord of the arc which contains three of the equal sides, $\sqrt{1 - x}$ will be the length of one side of a polygon of n sides described in a circle whose radius is *one*.

Let β represent the angle subtended by one of the equal sides of a polygon of n sides. Then is

$$x = 2 \cos \beta - 1. \quad \therefore \sqrt{1 - x} = \sqrt{2 - 2 \cos \beta}$$

$$= \sqrt{2} \sqrt{1 - \cos \beta} = \sqrt{2} \sqrt{1 - \sqrt{1 - \sin^2 \beta}} = \sqrt{1 + \sin \beta} - \sqrt{1 - \sin \beta}.$$

That is, one side of any regular polygon inscribed in a circle whose radius is *one*, is represented in functions of the sine of the arc subtended by that side, by

$$\sqrt{1 + \sin \beta} - \sqrt{1 - \sin \beta},$$

or by its development

$$\sin \beta + \tfrac{1}{8} \sin^3 \beta + \tfrac{25}{512} \sin^5 \beta + \&c. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By assigning any value to β, n will be determined. And if β be taken very small, n times (1) will represent approximately the circumference of a circle whose radius is one.

A very interesting relation exists between the isosceles triangles which are formed in the construction of polygons.

The triangle $A\ B\ C$ of the pentagon is divided into two triangles by the line $B\ D$. Multiply the area of the triangle $A\ B\ D$ by the line $B\ C$ and the product equals the area of the triangle $B\ D\ C$. Multiply the area of

this triangle by the line $B\ C$ and the product equals the area of the triangle $A\ B\ C$. That is, these three triangles form a geometrical series of which $B\ C$ is the ratio. The triangle $B\ D\ C$ is a mean proportional between the other two.

The triangle $A\ B\ I$ of the heptagon divides itself into three isosceles triangles; but only two of these together with the whole triangle form a geometrical series. Multiply the area of the triangle $A\ B\ R$ by $B\ I$ and the product will be equal to the area of the triangle $B\ R\ S$, and this last triangle multiplied by $B\ I$ will give the area of the triangle $A\ B\ I$. But the area of $A\ B\ R$ multiplied by $R\ I$ will give the area of $R\ S\ I$, and $R\ S\ I$ multiplied by $B\ S$ will give the area of $B\ R\ S$. Therefore, since $A\ B\ R \times R\ I \times B\ S = B\ R\ S$, $R\ I \times B\ S = B\ I$. And because $B\ H = A\ B$, therefore the triangle $A\ B\ H$ equals the triangle $B\ R\ S$.

The triangle $A\ B\ N$ of the nonagon divides itself into four isosceles triangles, but only two of these together with the whole triangle form a geometrical series.

Multiply the area $A\ B\ T$ by $B\ N$ and the product will be equal to the area $T\ h\ Q$, and this last multiplied by $B\ N$ will give the area of $A\ B\ N$. But the area of the triangle $A\ B\ T$ multiplied by $h\ N$ gives the area of the triangle $N\ Q\ h$, and the area of $A\ B\ T$ multiplied by $N\ T$ gives the area of the triangle $B\ T\ h$.

If the triangle of the polygon of eleven sides be constructed, the triangle will divide itself into five isosceles triangles and the third from the base will be the mean proportional. But in the triangle of the polygon of thirteen sides the fourth triangle from the base will be the mean proportional, there being six triangies formed in this case.

If in the pentagon whose sides each = *one* we put the chord which contains two of the equal sides $= 1 + x$, we get for the equation which determines the value of x,

$$x^2 + x = 1. \quad \ldots \ldots \ldots \quad (a)$$

And if in the heptagon and nonagon, whose sides = *one*, we put the chord which contains three of the equal sides $= 2 + x$, we get for the equations respectively, for the heptagon

$$x^3 + 4\,x^2 + 3\,x = 1, \quad \ldots \ldots \ldots \quad (b)$$

and for the nonagon

$$x^3 + 3\,x^2 = 1. \quad \ldots \ldots \ldots \quad (c)$$

And in general, we have, for the equation of a polygon of n sides, in which $2 + x$ represents the chord which contains three of the equal sides, (each

of which $=$ *one*), r the radius of the circumscribing circle, and β the angle subtended by one of the equal sides,

$$x = \frac{r^2-1}{r^2}, \text{ and } r = \frac{1}{\sqrt{2\,(1-\cos\beta)}}.$$

SOLUTION OF TWO INDETERMINATE PROBLEMS.

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Problem I.—Find n numbers in arithmetical progression, such that the sum of their cubes shall be a square number.

Assume the n terms of the progression as follows:

$$x-\frac{n-1}{2}d;\ x-\frac{n-3}{2}d;\ \ldots.x-\tfrac{1}{2}d;\ x+\tfrac{1}{2}d;\ \ldots.x+\frac{n-3}{2}d;\ x+\frac{n-1}{2}d,$$

which corresponds to the case when n is an even number, and

$$x-\frac{n-1}{2}d;\ x-\frac{n-3}{2}d;\ \ldots.x-d;\ x;\ x+d;\ \ldots.x+\frac{n-3}{2}d;\ x+\frac{n-1}{2}d,$$

which corresponds to the case when n is an odd number.

The sum of the cubes of the n terms, when n is even, is

$$n\,x^3 + \tfrac{3}{2}\,[1^2 + 3^2 + 5^2 + \ldots.(n-3)^2 + (n-1)^2]\,d^2\,x.$$

When n is odd, the sum of the cubes is

$$n\,x^3 + 6\,[1^2 + 2^2 + 3^2 + \ldots.(\tfrac{1}{2}\,\overline{n-3})^2 + (\tfrac{1}{2}\,\overline{n-1})^2]d^2\,x.$$

Each of these expressions, when simplified, becomes

$$n\,x\,[x^2 + (n^2-1)\,(\tfrac{1}{2}\,d)^2] \quad . \; . \; . \; . \; . \; . \; . \quad (a)$$

and this must be a square number.

$$\text{Assume } x^2 + (n^2-1)\,(\tfrac{1}{2}\,d)^2 = 4\,n\,x\,t^2,$$

and the above will become $4\,n^2\,x^2\,t^2 =$ a square.

$$\text{Solving } x^2 + (n^2-1)\,(\tfrac{1}{2}\,d)^2 = 4\,n\,x\,t^2$$

for x, we find

$$x = 2\,n\,t^2 + \sqrt{4\,n^2\,t^4-(n^2-1)\,(\tfrac{1}{2}\,d)^2} \quad . \; . \; . \; . \; . \quad (b)$$

Hence, in order that this value of x may be rational, we must make

$$4\,n^2\,t^4 - (n^2-1)\,(\tfrac{1}{2}\,d)^2 = \text{a square} = [2\,n\,t^2 - s\,(\tfrac{1}{2}\,d)]^2$$

$$\text{This gives } d = \frac{8\,n\,s\,t^2}{s^2+n^2-1}, \text{ and then}$$

$$x = \frac{4\,n\,(n^2-1)\,t^2}{s^2+n^2-1}.$$